1 Firstly, the matrix that will rotate the plane by 90° clockwise is given by

$$\begin{bmatrix} \cos(-90^\circ) & -\sin(-90^\circ) \\ \sin(-90^\circ) & \cos(-90^\circ) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

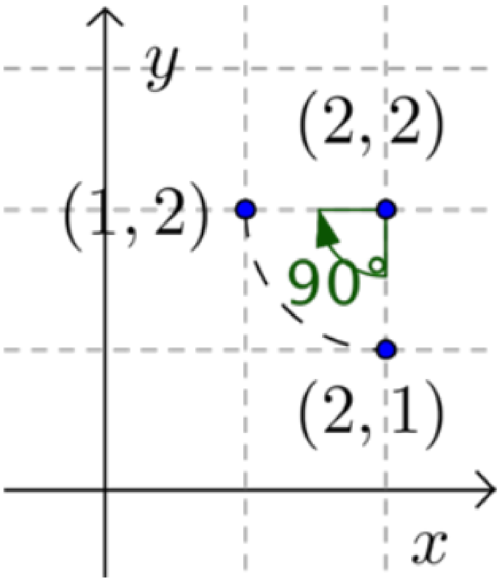
Therefore, the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x-2 \\ y-2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} y-2 \\ -x+2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} y \\ -x+4 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(2,1)$. Let $x=2$ and $y=1$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ -2+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore, $(2,1) \to (1,2)$, as expected from the diagram shown below.



2 Firstly, the matrix that rotates the plane by 180° about the origin is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

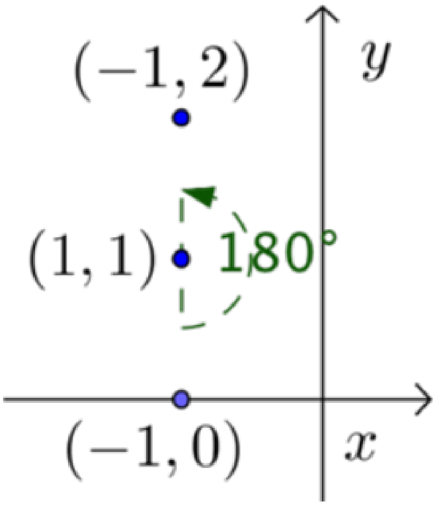
Therefore the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x+1 \\ y-1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -x-1 \\ -y+1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -x-2 \\ -y+2 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(-1,0)$. Let $x=-1$ and $y=0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -(-1)-2 \\ -0+2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Therefore, $(-1,0) \to (-1,2)$, as expected from the diagram shown below.



3 a Firstly, the matrix that reflects the plane in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

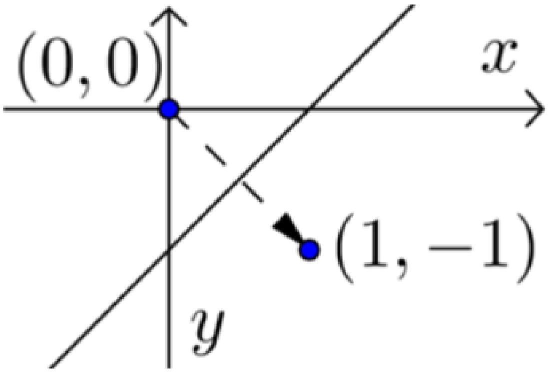
Therefore, the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} y+1 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(0,0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore, $(0,0) \to (1,-1)$, as expected from the diagram shown below.



b Firstly, the matrix that reflects the plane in the line $y = -x$ is given by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

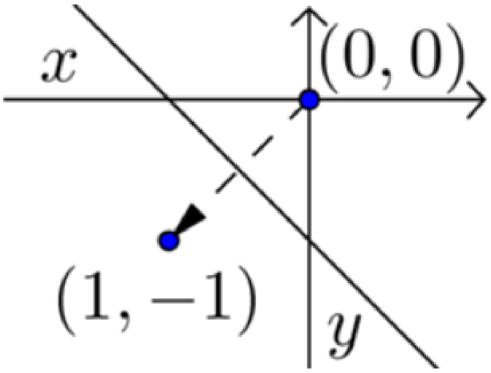
Therefore, the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y-1 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y-1 \\ -x-1 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0-1 \\ -0-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Therefore, $(0, 0) \to (-1, -1)$, as expected from the diagram shown below.



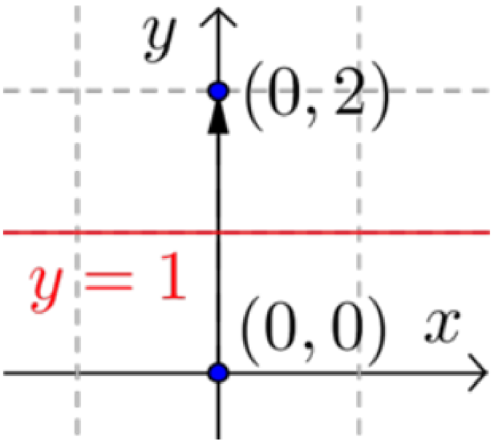
c We will translate the plane 1 unit down so that we can then reflect the plane in the line $y = 0$, that is, the x-axis. We then return the plane to its original position by translating the plane 1 unit up.
Therefore, the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -y+2 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ -0+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Therefore, $(0, 0) \to (0, 2)$, as expected from the diagram shown below.



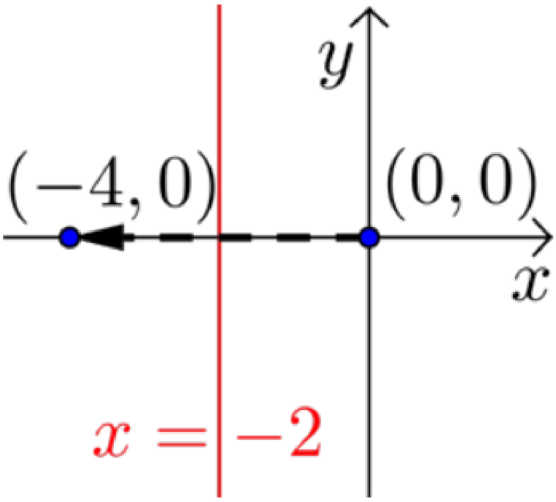
d We will translate the plane 2 units right so that we can then reflect the plane in the line $x = 0$, that is, the y-axis. We then return the plane to its original position by translating the plane 2 units left.
Therefore, the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x+2 \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -x-2 \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -x-4 \\ y \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0-4 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

Therefore, $(0, 0) \to (-4, 0)$, as expected from the diagram shown below.



4 We will rotate the plane clockwise by angle θ, dilate the point (x, y) by a factor of k from the y-axis, then return the plane to its original position by rotating by angle θ anticlockwise.
Therefore, the required matrix will be

$$\begin{aligned} &\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + k\sin^2\theta & \cos\theta\sin\theta - k\cos\theta\sin\theta \\ \cos\theta\sin\theta - k\cos\theta\sin\theta & \sin^2\theta + k\cos^2\theta \end{bmatrix}. \end{aligned}$$

5 We will rotate the plane clockwise by angle θ, project the point (x, y) onto the x-axis, then return the plane to its original position by rotating by angle θ anticlockwise.
Therefore, the required matrix will be

$$\begin{aligned} &\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}. \end{aligned}$$

6 The transformation that reflects the plane in the line $y = x + 1$ is given by,

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} y-1 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} y-1 \\ x+1 \end{bmatrix}. \end{aligned}$$

If we then want to reflect the result in the the line $y = x$ we would multiply by the reflection matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This gives a transformation whose rule is

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y-1 \\ x+1 \end{bmatrix} \\ &= \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}.\end{aligned}$$

This corresponds to a translation defined by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.